

# BRANCHED CYCLIC COVERS AND FINITE TYPE INVARIANTS

ANDREW KRICKER

**ABSTRACT.** This work identifies a class of moves on knots which translate to  $m$ -equivalences of the associated  $p$ -fold branched cyclic covers, for a fixed  $m$  and any  $p$  (with respect to the Goussarov-Habiro filtration). These moves are applied to give a flexible (if specialised) construction of knots for which the Casson-Walker-Lescop invariant (for example) of their  $p$ -fold branched cyclic covers may be readily calculated, for any choice of  $p$ .

In the second part of this paper, these operations are illustrated by some theorems concerning the relationship of knot invariants obtained from finite type three-manifold invariants, via the branched cyclic covering construction, with the finite type theory of knots.

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## 1. INTRODUCTION

Let  $K$  be the free abelian group generated by (oriented) equivalence classes of pairs  $(M, K)$ , where  $M$  is an integral homology three-sphere and  $K$  is an oriented knot in it. Let  $M$  be the free abelian group generated by homeomorphism classes of three-manifolds. Let the mapping

$$\Sigma^p : K \rightarrow M$$

be the linear extension of the operation defined on a pair  $(M, K)$  as the  $p$ -fold branched cyclic covering of  $M$  branched over  $K$ . We are chiefly interested in knots in the three-sphere, for which the corresponding space will be denoted  $K(S^3)$ , when required.

The subject of this investigation is the set of knot invariants obtained by composing the branched cyclic cover construction, for a fixed degree, with finite-type invariants of three-manifolds (of various descriptions).

This paper is in two parts. The main theorem of the first part is Theorem 2.0.5, and its enhancement, Theorem 2.0.8. This theorem describes the effect of surgery on **complete graph Y-links** on the  $p$ -fold branched cyclic covers of a knot, with respect to the Goussarov-Habiro filtration. A calculus is developed in Lemmas 2.0.11 and 2.0.12 which describes this action.

In the second part of the paper, we illustrate this calculus by exploring some very general questions about the invariants obtained by composing projections of the LMO invariant with the operation of taking a  $p$ -fold branched cyclic cover. The main theorem of this part is the realisation theorem, Theorem 4.0.4, which has the following, possibly unsurprising corollary:

**Corollary 1.0.1.** *Take a positive integer  $p$ . If  $v$  is a rational valued three-manifold invariant factoring non-trivially through the LMO invariant on integral homology spheres, then  $v \circ \Sigma^p$  is not a finite-type invariant of knots.*

It is interesting that we can use a finite-type property (in one sense) to show that an invariant is not finite-type (in some different sense).

The generality of the above theorem does not seem to follow from the more descriptive investigations of specific examples and projections of the LMO invariant that have been recorded: Hoste gave a formula for the Casson invariant of the  $p$ -fold branched cyclic cover over an untwisted double of a knot [Hos]; Davidow gave formulae for the case of iterated torus knots [Dav1], and for some 1-twisted doubles [Dav2]; Mullins calculated the composition of the Casson-Walker invariant with the 2-fold branched cyclic cover in terms of the Jones polynomial, when the left hand side was well-defined [Mul1, Mul2]; Ishibe used this work to give formulae for the general case of  $m$ -twisted doubles [Ish]; Garoufalidis showed that Mullins formulae was valid in general, using the Casson-Walker-Lescop invariant [Gar]; and also in that paper Garoufalidis initiated the investigation into the LMO invariant on branched cyclic covers, describing the form of the result for covers over twisted doubles.

The main motivation for this work was to see whether the techniques we employed in [K] (an application of clasper theory) had wider application. The other main motivation was to explore a possible approach to Lev Rozansky's program for a "finite type theory of knots' complements" [Roz]. We expect the chief interest in this article to be the analysis of a suggestive family of operations which will be important in the development of a more structural relationship between the

loop expansion of the Kontsevich integral, and the degree expansion of  $Z^{LMO} \circ \Sigma^p$ . (That is the informal conjecture we are seeking to motivate with this paper.)

A review of some of the theory of Goussarov-Habiro filtration is included as an appendix.

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## 2. PART 1: COMPLETE CLASPER MOVES AND BRANCHED CYCLIC COVERS

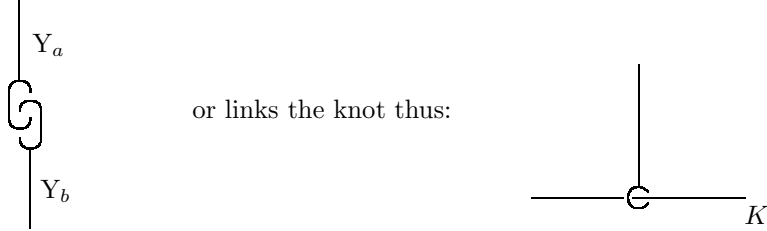
The Goussarov-Habiro filtration [G3, Hab] is a descending filtration of  $\mathbf{M}$  (see Definition A.0.2):

$$\mathbf{M} \supset \mathbf{F}_1^Y \mathbf{M} \supset \mathbf{F}_2^Y \mathbf{M} \supset \dots$$

On the other hand we have clasper calculus, introduced by Habiro [HT, Hab], certain aspects of which were independently developed by Goussarov [G2]. This generalised Matveev’s “Borromeo” move [Mat] and Murakami-Nakanishi’s “delta-unknotting” operation [MN].

In this work, we will identify a subclass of clasper operations which the branched cyclic covering construction will translate to  $n$ -equivalences of three-manifolds. For technical reasons, this work will use the language of Y-links; note that our language is slightly different to that adopted in [GL2].

In Appendix A we will fix the term *graph Y-link* to mean a Y-link decoration of a knot  $K$  such that each leaf of every Y-component links either another leaf, in a ball (see Appendix A for diagrammatic conventions):



Note that we also consider graph Y-links which may be *without legs*. Note that surgery on a graph Y-link without legs will change the homeomorphism class of the ambient manifold; surgery on a connected graph Y-link with at least one leg will not.

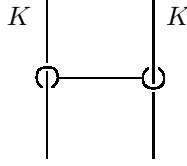
To every graph Y-link there is canonically associated (up to sign) a uni-trivalent diagram (which here refers to the familiar uni-trivalent graphs, which may have univalent vertices located on a skeleton; when we take linear combinations, we introduce STU, IHX and AS relations). This association is made by replacing Y-components by trivalent vertices, replacing leaves linking the knot by legs, and by joining tines of trivalent vertices by an edge when the corresponding leaves were linked. A graph Y-link is called *connected* when the dashed graph of this associated

diagram is connected. The *degree* of a graph Y-link is half the number of vertices of the dashed graph of the associated uni-trivalent diagram.

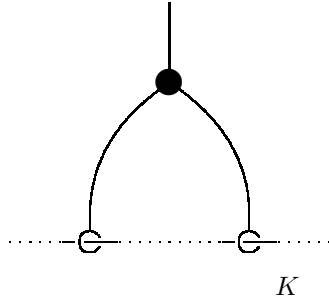
To introduce the moves in question, some definitions are required.

**Definition 2.0.3.** A graph Y-link decorating a knot  $K$  is *complete* if it is connected and if every leg of the associated uni-trivalent diagram leads to a separate trivalent vertex.

In particular, we are ruling out the chord:



and graph Y-links which end in a fork:



It is important to point out that this is quite a strong restriction: for example, a move of such a kind on a link will not affect its Milnor invariants.

Now we need an alternative measure on graph Y-links.

**Definition 2.0.4.** The *surplus* of a uni-trivalent diagram is defined to be the number of trivalent vertices minus the number of univalent vertices of its dashed graph. The *surplus* of a graph Y-link is defined to be the surplus of its associated uni-trivalent diagram.

For example, the surplus of a complete graph Y-link is simply the grade of the underlying uni-trivalent diagram, after all the legs have been removed. The main theorem is expressed in these terms.

**Theorem 2.0.5** (Main theorem). *If a knot  $K^L$  is obtained from a knot  $K$  by surgery on a complete graph Y-link  $L$  of surplus  $s$ , where  $s \geq 2$ , then*

$$\Sigma_{K^L}^p - \Sigma_K^p \in \mathbb{F}_s^Y \mathbb{M}.$$

There is another important invariant for which a similar property holds. The following theorem [KGr], and the theory of which it is an application, may appear in a future manuscript.

**Theorem 2.0.6.** *If a knot  $K^L$  is obtained from a knot  $K$  by surgery on a complete graph Y-link  $L$  of surplus  $s$ , where  $s \geq 2$ , then*

$$Z^{LMO}(K^L) - Z^{LMO}(K) = \{\text{series of diagrams of surplus } \geq s\},$$

where  $Z^{LMO}$  is the Kontsevich-LMO (Le-Murakami-Ohtsuki) invariant of knots in integral homology three-spheres.

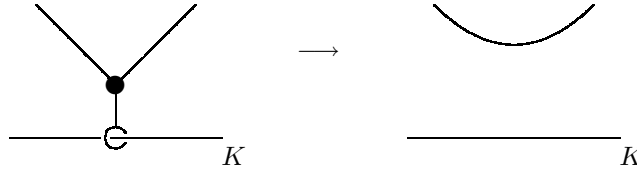
The point being that it is true for every grade. We leave the reader to make their own conjectures.

Returning to the main theorem of this paper, the next natural question is how these moves operate on the leading term in the space of associated graded quotients. Specifically, what is

$$\Sigma_{K^L}^p - \Sigma_K^p \in \mathcal{G}_s^Y \mathcal{M} ?$$

Lemma 2.0.11 and Lemma 2.0.12 calculate this term. To report this calculation, let the ambient integral-homology three-sphere be presented by some surgery link, and isotope  $K$  and the decorating graph Y-link in that three-manifold to appear in the complement of that surgery link in the three-sphere.

**Definition 2.0.7.** Let  $\lambda_L$  be a graph Y-link in  $S^3$  obtained by sawing off the legs of  $L$



and then forgetting  $K$  and the surgery link.

Let  $\lambda'_L$  be the corresponding graph Y-link in  $\Sigma_K^p$  induced by the connect-sum of that three-sphere into  $\Sigma_K^p$ . Let  $(\Sigma_K^p)^{\lambda'_L}$  denote the result of surgery on that graph Y-link.

Lemma 3.2.1 asserts that  $(\Sigma_K^p)^{\lambda'_L} - \Sigma_K^p$  is well-defined in  $\mathcal{G}_s^Y \mathcal{M}$ . Lemma 2.0.11 and Lemma 2.0.12 show that:

**Theorem 2.0.8** (Main theorem, part 2).

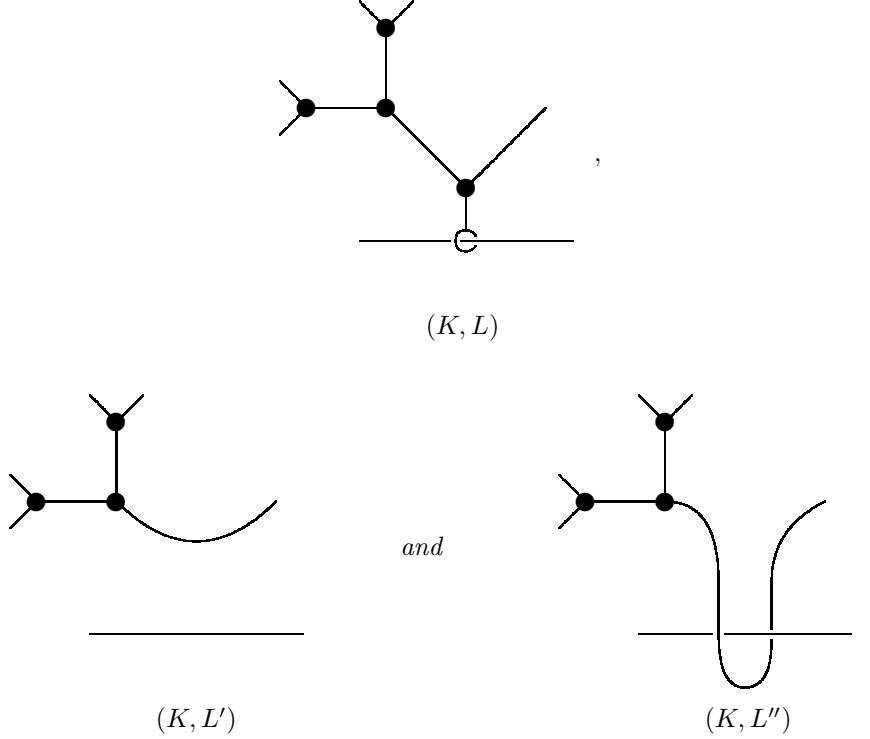
$$\Sigma_{K^L}^p - \Sigma_K^p \propto (\Sigma_K^p)^{\lambda'_L} - \Sigma_K^p \in \mathcal{G}_s^Y \mathcal{M}.$$

*Remark 2.0.9.* The definition of  $(\Sigma_K^p)^{\lambda'_L} - \Sigma_K^p$  is perhaps more easily understood in terms of the surjective map from the space of marked uni-trivalent graphs of degree  $s$  to  $\mathcal{G}_s^Y \mathcal{M}([M])$ . In this sense, the first non-vanishing term is proportional to the image under that map of the diagram obtained by removing all the legs from a diagram associated to the initial decoration. The procedure we employ in this paper is used so that we can be precise, in a natural way, about the sign of the element referred to (see Section 3.1).

The following two lemmas can be used to immediately calculate this term. Calculation proceeds in two steps. The first lemma is used to reduce the number of legs on  $K$ , so that eventually  $\Sigma_{K^L}^p - \Sigma_K^p$  is expressed as a linear combination of differences  $\Sigma_{K_i}^p - \Sigma_K^p$  where each  $K_i$  has been obtained from  $K$  by surgery on a connected graph Y-link without legs. The second lemma calculates an expression for the terms in that sum.

*Remark 2.0.10.* This procedure is the reason it is more natural in this context to work in the generality of integral homology three-spheres. It is interesting that the calculation necessitates this extension, not unlike the results of [GL2].

**Lemma 2.0.11.** *Let  $K$  be a knot. Let  $L$ ,  $L'$  and  $L''$  denote three decorations of  $K$  by complete graph  $Y$ -links of surplus  $s$  that differ in a ball in the following way:*



Then

$$(\Sigma_{K^L}^p - \Sigma_K^p) = (\Sigma_{K^{L'}}^p - \Sigma_K^p) - (\Sigma_{K^{L''}}^p - \Sigma_K^p) \in G_s^Y M.$$

The statement of the next lemma requires an embedding of a graph in  $M - K$  that will be associated to a decoration of  $K$  by a graph  $Y$ -link  $L$  in the complement of  $K$ . This embedding is constructed by contracting the clasper graph presenting  $L$  onto a spine: call this embedded graph  $E_L$ . In the statement of the lemma, a loop on  $L$  will mean precisely a path in  $M - K$  that corresponds to a loop on  $E_L$ .

**Lemma 2.0.12.** *Let a knot  $K^L$  be obtained from a knot  $K$  by surgery on some connected graph  $Y$ -link  $L$  without legs of surplus  $s$ . Then*

$$\Sigma_{K^L}^p - \Sigma_K^p = \begin{cases} p((\Sigma_K^p)^{\lambda_L} - \Sigma_K^p) & \text{if every loop on } L \text{ has zero linking} \\ & \text{number with } K \text{ modulo } p, \\ 0 & \text{otherwise.} \end{cases} \in G_s^Y M.$$

Intuitively speaking, this is what one would expect. The implementation of that intuition in this setting, however, is not immediate (Section 3.4).

Let us illustrate this calculus with some examples based on the most familiar Goussarov-Habiro finite type invariant, the Casson-Walker-Lescop invariant, denoted  $\lambda_{CWL}$ . The connection will be made with the following lemma. Extend the symbol  $|H_1(M)|$  to be zero in the case that  $M$  has non-vanishing first Betti number.

**Lemma 2.0.13.**

1.  $\lambda_{CWL}(G_3^Y M) = 0$ .
2. *Let some three-manifold  $M$  have some graph  $Y$ -link  $\Theta$  embedded in it, whose underlying diagram is the “theta” graph. Then*

$$\lambda_{CWL}(M^\Theta) = \lambda_{CWL}(M) \pm 2|H_1(M)|,$$

where the sign can be determined in a given situation.

Perhaps the easiest way to see this is to observe the identification of the coefficient of the theta graph in the LMO invariant with half the Casson-Walker-Lescop invariant, [LMMO] (see also [GH, HB, L]), and then observe that the difference of the LMO invariant on a manifold and on that manifold surgered along a  $\Theta$  is precisely  $\pm|H_1(M)|\theta$  plus higher order terms, where the sign can be determined in a given situation (see [LeGr] for such first non-vanishing term calculations).

**Example 2.0.14.** Fix an integer  $p$  and take a knot  $K$ . Consider some graph  $Y$ -link decoration  $\Theta$  of  $K$  with underlying diagram the “theta” graph, and let  $K^\Theta$  be the knot (in some integral homology three-sphere) obtained from surgery on  $\Theta$ . Let  $\theta$  denote the spatial graph in  $S^3 - K$  associated to this decoration. Let  $l_1$  denote a choice of a knot in  $S^3 - K$  obtained by forgetting some edge of  $\theta$ , and let  $l_2$  denote a knot obtained by forgetting some other choice of edge. Then:

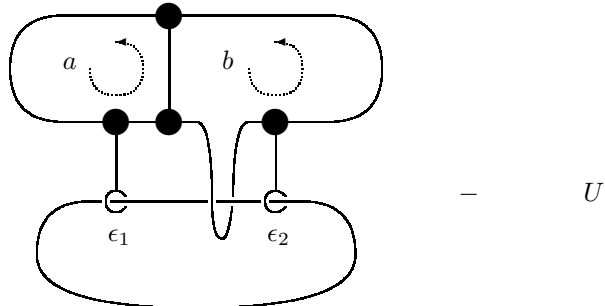
$$\lambda_{CWL}(\Sigma_{K^\Theta}^p) = \lambda_{CWL}(\Sigma_K^p) \pm \begin{cases} 2p|H_1(\Sigma_K^p)| & \text{if } \text{lk}(l_i, K) = 0 \bmod p, i = 1, 2, \\ 0 & \text{otherwise,} \end{cases}$$

where the sign can be determined in a given situation. This follows from Lemma 2.0.12.

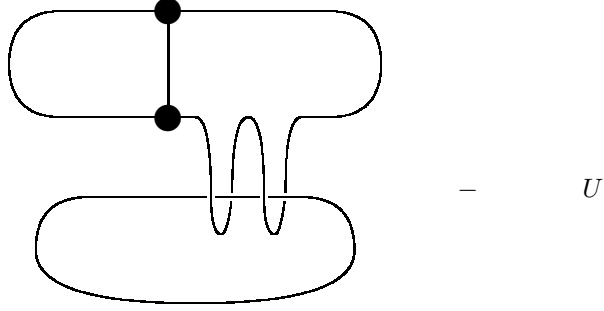
□

How might we organise such a calculation in generality? That is, where we have a decoration of a knot by a theta graph with some legs attached. Such an example follows.

**Example 2.0.15.** We indicate a basis  $\{a, b\}$  for the first homology of the associated trivalent graph, and some labels  $\{\epsilon_1, \epsilon_2\}$  have been affixed to the legs. Let  $U$  indicate the unknot, and let the decorating graph  $Y$ -link be denoted by  $\kappa$ .



Theorem 2.0.5 indicates that under  $\Sigma^p$  this difference lies in  $G_s^Y \mathbf{M}$ . To calculate the term we proceed by applying Lemma 2.0.11 to each of the legs. This produces four terms, which will be recorded by setting each of the  $\epsilon_i$  to either 0 or 1. For example the term that corresponds to  $(\epsilon_1, \epsilon_2) = (0, 1)$  will be:



Now, according to Lemma 2.0.12, precisely those terms for which the cycles  $a$  and  $b$  have mod- $p$  linking number zero with  $U$  will contribute. For a given  $(\epsilon_1, \epsilon_2)$ , the linking numbers will be (giving  $U$  the clockwise orientation):

$$\begin{aligned} lk(a, U) &= \epsilon_1, \\ lk(b, U) &= \epsilon_2 + 1. \end{aligned}$$

If we collect this information into the monomial  $F_{\epsilon_1, \epsilon_2}(t_a, t_b) = t_a^{lk(a, U)} t_b^{lk(b, U)}$ , then for a given  $p$ , for some determinable sign  $\alpha$ :

$$\lambda_{CWL} = \alpha 2p \sum_{(\epsilon_1, \epsilon_2) = (0, 0)}^{(1, 1)} \left( \frac{1}{p} \sum_{r=0}^{p-1} \frac{1}{p} \sum_{s=0}^{p-1} F_{\epsilon_1, \epsilon_2}(e^{r \frac{2\pi i}{p}}, e^{s \frac{2\pi i}{p}}) \right).$$

[ To see this, note the following identity:

$$\sum_{s=0}^{p-1} (e^{\frac{2\pi i}{p} s})^t = \begin{cases} p & \text{if } p|t, \\ 0 & \text{otherwise.} \end{cases}$$

We can separate out the dependence on  $p$  as follows. Defining the polynomial:

$$F'(t_a, t_b) = \sum_{(\epsilon_1, \epsilon_2) = (0, 0)}^{(1, 1)} F_{\epsilon_1, \epsilon_2}(t_a, t_b),$$

then the result is:

$$\lambda_{CWL}(\Sigma_{U^\kappa}^p) = \alpha \frac{2}{p} \sum_{r=0}^{p-1} \sum_{s=0}^{p-1} F'(e^{r \frac{2\pi i}{p}}, e^{s \frac{2\pi i}{p}}).$$

*Remark 2.0.16.* Observe that such a procedure can be implemented for any decoration of any knot by a complete graph  $Y$ -link whose underlying diagram is a theta graph with some legs attached. Thus we have a flexible new construction of knots for which the Casson-Walker-Lescop invariants of their  $p$ -fold branched cyclic covers may be calculated, for all choices of  $p$ . It is interesting that this sequence is determined by the values of an associated polynomial at the roots of



unity, evoking the familiar formula for the order of the first homology in terms of the Alexander-Conway polynomial [F, HK] of the branching knot.

Observe also that similar constructions and arguments can be applied to any projection of the LMO invariant (see Example 4.1.2).

Section 2.1 will prove part 1 of the main theorem. Section 3 will prove the Lemmas 2.0.11 and 2.0.12 which facilitate the extension.

**2.1. Branched cyclic covers.** Excellent expositions of this construction can be found in the literature, for example in [Rol] and in [Lic].

Let  $N(K)$  be a regular neighbourhood of a knot  $K$  in an integral homology three-sphere  $M$ , and let  $X_K$  be the closure of its exterior. Let  $X_K^p$  be the  $p$ -fold cyclic cover of  $X_K$ .

In this work, it will be convenient to construct this space directly with a Seifert surface  $F$  for  $K$ . This direct construction uses the space  $W_F$ , which is obtained by “splitting  $X_K$  open along  $F$ ”. In practice,  $W_F$  is obtained by removing the intersection of an open collar neighbourhood of the surface  $F$  with  $X_K$ : that is, if  $N : F \times [-1, 1] \hookrightarrow M$  is a bicollar for  $F$  then  $W$  is obtained as the removal of  $X_K \cap N(F \times (-\epsilon, \epsilon))$  from  $X_K$ . The subspaces  $X_K \cap N(F \times \{\epsilon\})$  and  $X_K \cap N(F \times \{-\epsilon\})$  are homeomorphic images of  $F$ , call them  $F^+$  and  $F^-$ , and there will exist a homeomorphism between them  $\phi : F^- \rightarrow F^+$ , a gluing along which recovers the exterior of the knot,  $X_K$ .

The unique  $p$ -fold cyclic cover of the exterior of the knot,  $X_K^p$ , is constructed from  $p$  copies of  $W_F$ : denote them  $W_F^i$ , where the index  $i$  is an element of  $\mathbb{Z}_p$ . Glue  $F^-$  in  $W_F^i$  to  $F^+$  in  $W_F^{i+1}$  using  $\phi$  composed with the identification  $W_F^i \simeq W_F^{i+1}$ .

The  $p$ -fold cyclic cover of  $M$  branched over  $K$  is obtained as a completion of the space  $X_K^p$ . Specifically, a solid torus is glued into the torus boundary of  $X_K^p$  so that its meridian projects to a meridian of  $K$  traversed  $p$  times.

In this work, we will encounter a situation where a knot  $K^L$  is recovered from another knot  $K$  by surgery on some framed link  $L$  decorating  $K$ , and where there is given some Seifert surface  $F$  for  $K$  in the complement of  $L$ . In such a situation, a Seifert surface  $F^L$  for  $K^L$  may be canonically chosen: namely, just choose the position of  $F$  in the surgered manifold. The pair  $(W_{F^L}, \phi^L)$  can then be obtained from the pair  $(W_F, \phi)$  by surgery on the link  $L$  in  $W_F$ . Thus, by construction,  $\Sigma_{K^L}^p$  is recovered by surgery on the framed link in  $\Sigma_K^p$  comprising of one copy of the original framed link in each of the  $p$  copies of  $W_F$  included in  $\Sigma_K^p$ .

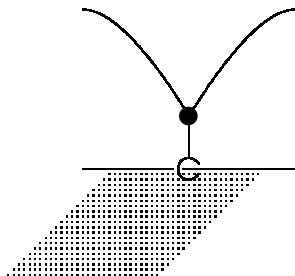
**2.2. Complete graph Y-links.** Before we describe this proof, we introduce a term.

**Definition 2.2.1.** A **mixed Y-link** in a three-manifold is an embedding of a set of Y-components *and possibly* some claspers, into that three-manifold.

Let us turn to the scenario described by the main theorem of this section. We have two knots:  $K$  and a knot  $K^L$  obtained from  $K$  by surgery on a complete graph Y-link  $L$  of surplus  $s$ . The strategy is to replace  $L$  with a mixed Y-link also presenting the knot  $K^L$ , but which is in the complement of a Seifert surface  $\tilde{F}$  for  $K$ .

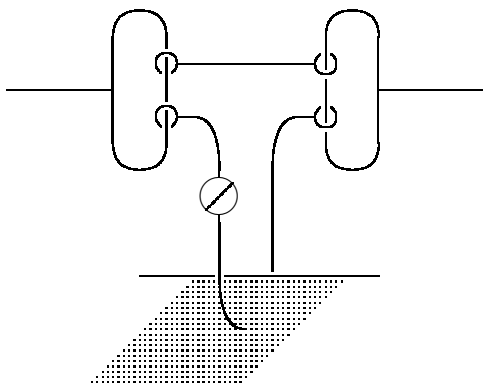
To begin, choose some Seifert surface  $F$ . We will now isotope the graph Y-link  $L$  so that it is in a special position with respect to  $F$ . We require two conditions of this position. The first requirement is that the band of a clasper associated to

a leg does not intersect  $F$  (although one of the leaves certainly will). Given such a position, which can always be obtained, then around every leaf we can find a ball so that the arrangement is obtained by some homeomorphism of that ball with the following ball in  $S^3$ :



The second requirement is that the only other intersections of  $L$  with  $F$  are transversal intersections of bands with  $F$ .

A graph Y-link in such a position will now be modified in two steps. The first step is make an adjustment of  $L$  around each leg, where it appears as in the ball above. The adjustment is made with the move described in Lemma A.0.5:



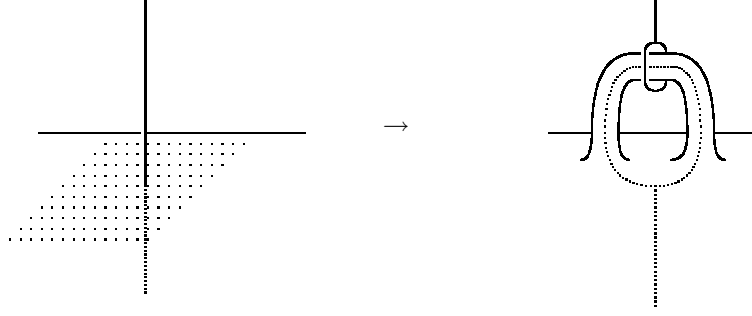
There is one such adjustment to be made for every (if there are any, that is) leg of the graph Y-link, leaving the decoration with one transverse intersection of a band with  $F$  for every leg, and with possible further transverse intersections of internal bands with  $F$ .

The second step is to adjust the resulting mixed graph Y-link *and Seifert surface* at each such intersection (so one for each leg with possible further adjustments). Using the move described in Lemma A.0.4 the band can be broken into a clasp at that intersection, then  $F$  can be tubed around the introduced annulus (this is shown in the diagram following the proof).

After these adjustments, we have a mixed graph Y-link in the complement of a Seifert surface for  $K$ . In this mixed graph Y-link, one can identify  $s$  Y-components (one for each surplus trivalent vertex) and a possible number of other clasps that

have emerged from the modifications. By an abuse of notation call this mixed Y-link  $L$ . This has been constructed to be in the complement of some Seifert surface for  $K$ , call it  $\tilde{F}$ .

Use this Seifert surface to construct  $\Sigma_K^p$ . Denote by  $\tilde{L}$  the Y-link comprised of a copy of  $L$  for each of the  $p$  copies of  $W_{\tilde{F}}$  included into  $\Sigma_K^p$ . By construction, surgery on this mixed Y-link yields  $\Sigma_{KL}^p$ .



### 2.3. Proof of Theorem 2.0.5.

Our task is to show that  $\Sigma_K^p$  and  $\Sigma_{KL}^p$  are  $s - 1$ -equivalent. It is sufficient, by Lemma A.1.3, to exhibit an  $s$ -scheme relating  $\Sigma_{KL}^p$  to  $\Sigma_K^p$ . This  $s$ -scheme will be based upon the mixed Y-link  $\tilde{L}$  in  $\Sigma_K^p$ .

The appropriate selection of Y-sublinks is immediate. Order the Y-components of  $L$ , denoting them  $Y_1$  through to  $Y_s$ . Let  $\tilde{L}_i$  denote the Y-sublink of  $\tilde{L}$  comprised of the set of  $p$  copies of  $Y_i$ . These  $s$  disjoint Y-sublinks form the required relating  $s$ -scheme:  $\{\tilde{L}; \tilde{L}_1, \dots, \tilde{L}_s\}$ . Recall that  $\tilde{L}_{i_1, \dots, i_s}$  denotes the link obtained by forgetting those sublinks  $\tilde{L}_j$  for which  $i_j = 1$ . By construction, the result of surgery on  $\tilde{L}_{i_1, \dots, i_s}$  is the  $p$ -fold branched cyclic cover branched over the knot obtained from  $K$  by doing surgery on  $L_{i_1, \dots, i_s}$ . If this multiplet is not  $\{0, \dots, 0\}$  then this knot will be  $K$ .

□

**2.4. A convenient expression.** Lemma A.1.4 can be employed to give an expression for  $\Sigma_{KL}^p - \Sigma_K^p$  that will be employed frequently in the sequel. Let  $\tilde{L}^{(a_1, \dots, a_s)}$  denote the sublink of  $\tilde{L}$  that forgets, for each  $i$  between 1 and  $s$ , every copy of  $Y_i$  except the copy in the included subspace  $W_{\tilde{F}}^{a_i}$ .

**Lemma 2.4.1.**

$$\Sigma_{KL}^p - \Sigma_K^p = \sum_{(a_1, \dots, a_s) = (1, \dots, 1)}^{(p, \dots, p)} [\Sigma_K^p, \tilde{L}^{(a_1, \dots, a_s)}].$$

## 3. PROOFS OF THE LEMMAS

Before we prove these lemmas, we must first recollect how to compare two graph Y-links in a given three-manifold.

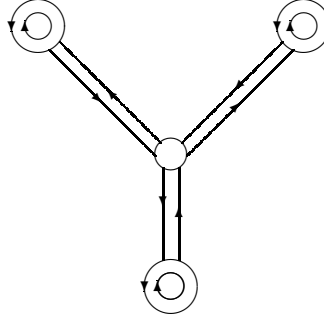
**3.1. Comparing graph Y-links in three-manifolds.** To an  $n$ -component graph Y-link  $L$  in a three-manifold  $M$  is canonically associated a trivalent graph  $D_L$ : take a trivalent vertex for every Y-component, and join them with an edge when the associated leaves are linked in  $M$ . Note that, at this point, we are not associating any extra structure to this diagram (like cyclic orderings of edges at trivalent vertices, or orientations of edges). Let  $L'$  be another  $n$ -component graph Y-link and let there exist a graph isomorphism between their associated diagrams  $D_L$  and  $D_{L'}$ .

**Lemma 3.1.1.**

$$[M, L] = \pm [M, L'] \in G_n^Y M.$$

Proof. Take some surgery link presenting  $M$ . The clasper graphs presenting the Y-links  $L$  and  $L'$  can be isotoped in  $M$  to appear in the complement of the surgery presentation in  $S^3$ . We now “move” the position of  $L'$  to that of  $L$ . That is, first isotope the vertices (discs of the clasper graph) of  $L'$  (in the complement of the surgery link) to occupy the same position as the corresponding vertices of  $L$ , choosing the identification of two discs which aligns the appropriate edges. Now  $L'$  can be adjusted to be in the position of  $L$  by crossing changes between its internal bands, and by crossing changes between its internal bands and surgery components (using Lemma A.0.7). A number of half-twists will possibly be introduced as the last step (using Lemma A.0.2). □

In this work, we will employ the following procedure to determine the sign of the above comparison. Begin by choosing an orientation for each Y-component of  $L$  (that is, orient the associated thrice-punctured disc). This induces an orientation on the associated boundary components, which will appear as follows (or its mirror image):



A graph Y-link with such a choice at every Y-component will be referred to as an *oriented graph Y-link*, and the choice will be called an *orientation* for it. This choice is equivalent to a choice of cyclic ordering of the leaves at every Y-component. To fix this correspondence, for example, say that the edges of the above Y-component have a counter-clockwise cyclic ordering. Record this information on the associated graph (that is, cyclically order the edges at every trivalent vertex of it). Call this enhancement an *orientation* for the graph.

To each edge of an oriented graph, associated to an oriented graph Y-link, a sign will be associated: the *twist* of that edge. The twist is defined as the linking number

of the (oriented) inner circles of the two leaves whose linking that edge represents. Let  $T(e)$  denote the twist of the edge  $e$ .

This information can now be used to compare two graph Y-links,  $L$  and  $L'$ , equipped with an isomorphism  $\phi : D_L \rightarrow D_{L'}$ . Namely, choose an orientation for  $L$ , inducing an orientation on  $D_L$ . The graph isomorphism induces an orientation on  $D_{L'}$ , and hence on  $L'$ .

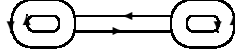
**Lemma 3.1.2.**

$$[M, L] = \left( \prod_{e \text{ an edge}} T(e)T'(\phi(e)) \right) [M, L'] \in \mathbf{G}_n^Y \mathbf{M}$$

Proof. We enhance the proof of Lemma 3.1.1. Notice that the crossing changes that are made as  $L'$  is “moved” to  $L$  do not affect the twist of any edge. Introducing a half twist, however, affects the twist of the associated edge by multiplying it by minus one. □

It will also prove necessary, in the sequel, to have an expression for the twist of an edge whose associated pair of leaves are linked indirectly by a chain of claspers. That is, where a leaf  $L_i$  of a Y-component  $Y_i$  is linked in a ball with a leaf of a clasper  $C_1$ , whose other leaf links in a ball with a leaf of another clasper  $C_2$ , and so on, until the clasper  $C_\mu$  whose other leaf links a leaf  $L_f$  of the other Y-component  $Y_f$ .

To calculate the twist in such a situation (that is, the twist that results from surgeries on all those claspers), begin by choosing an orientation for each clasper in the chain (that is, an orientation for the twice-punctured disc). It will appear as below (or its mirror image):



Let  $l_0$  be the linking number of the inner circles of the annuli corresponding to the appropriate pair of leaves, one from  $L_i$  and one from  $C_1$ ; let  $l_1$  be the linking number of the inner circles of the appropriate pair of leaves, one from  $C_1$  and one from  $C_2$ ; and so on. The following fact is straightforward:

**Lemma 3.1.3.** *The following product is well-defined (independent of the choices made in orienting the claspers) and*

$$T(e) = l_0 \prod_{i=0}^{\mu} (-l_i).$$

*Remark 3.1.4.* Actually, this is more information than we will use. All we need to know is that the resulting twist is a function of the linking numbers along the chain. One can prove this fact by considering the effect on the left-most linking number of a surgery on the left-most clasper; and proceed inductively.

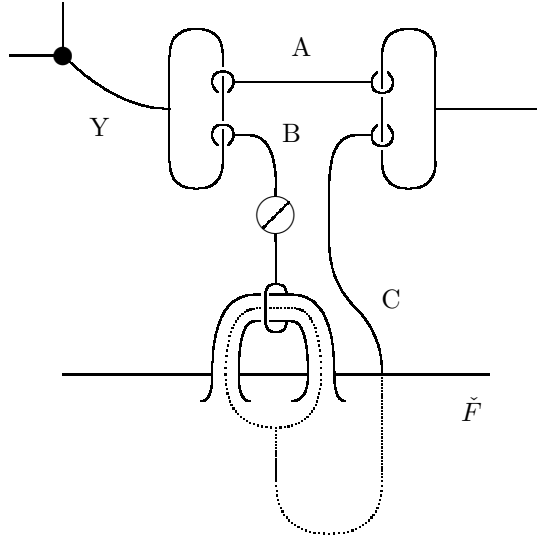
### 3.2. Proof of Lemma 3.2.1.

**Lemma 3.2.1.**  $(\Sigma_K^p)^{\lambda'_L} - \Sigma_K^p$  is well-defined in  $G_s^Y \mathbf{M}$ .

Proof. If the Y-components of  $L$  are oriented (according to the Section 3.1), then any graph Y-links in  $\Sigma_K^p$  resulting from the definition of  $\lambda'_L$  will have isomorphic underlying vertex-oriented graphs with identical twists along the edges, so that the well-definedness follows from Lemma 3.1.2. □

### 3.3. Proof of Lemma 2.0.11.

The graph Y-link  $L$  is modified in the ball, following the manoeuvres of Section 2.2, to appear in that ball in the following way:



Surgery on this mixed Y-link recovers the knot  $K^L$ . The knot  $K^{L'}$  is recovered by surgery on the sublink that is obtained by forgetting the claspers B and C, and the knot  $K^{L''}$  is obtained by surgery on the subgraph that is obtained by forgetting the clasper A and smoothing the half-twist. By an abuse of notation, we will refer to these mixed Y-links as  $L$ ,  $L'$  and  $L''$ .

The branched cyclic covering  $\Sigma_K^p$  (resp.  $\Sigma_{K^{L'}}^p$ , resp.  $\Sigma_{K^{L''}}^p$ ) is obtained from  $\Sigma_K^p$  by performing surgery on the mixed Y-link that is comprised of a copy of  $L$  (resp.  $L'$ , resp.  $L''$ ) for each of the  $p$  subspaces  $W_{\tilde{F}}$  included in  $\Sigma_K^p$ . Call these mixed Y-links  $\tilde{L}$ ,  $\tilde{L}'$  and  $\tilde{L}''$ . The mixed Y-link  $\tilde{L}$  will have  $p$  copies of each of the components illustrated above. This set of components will be denoted by affixing a  $\mathbb{Z}_p$ -valued superscript to the labels Y, A, B and C. We will use the same symbols to denote the corresponding components in the sublinks  $\tilde{L}'$  and  $\tilde{L}''$ .

To fix this notation fully, it remains to specify which boundary component after the removal of  $\tilde{F}$  corresponds to  $\tilde{F}^+$ . This choice can be specified if we declare that we are making the choice that results in  $C^i$  linking  $B^{i+1}$  in a ball, as follows:



Order the Y-components of  $L$  so that the one pictured above is  $Y_1$ . Let  $\tilde{L}^{(a_1, \dots, a_s)}$  denote the sublink of  $\tilde{L}$  that is obtained by forgetting every Y-component, except the copy of  $Y_i$  in  $W_{\tilde{F}}^{a_i}$ , for each  $i$ . Similarly use the symbol  $\tilde{L}'^{(a_1, \dots, a_s)}$  (resp.  $\tilde{L}''^{(a_1, \dots, a_s)}$ ) for the sublink obtained from  $\tilde{L}^{(a_1, \dots, a_s)}$  by forgetting every  $B^i$  and  $C^j$  (resp. forgetting every  $A^k$  and smoothing the half-twist in every  $B^l$ ).

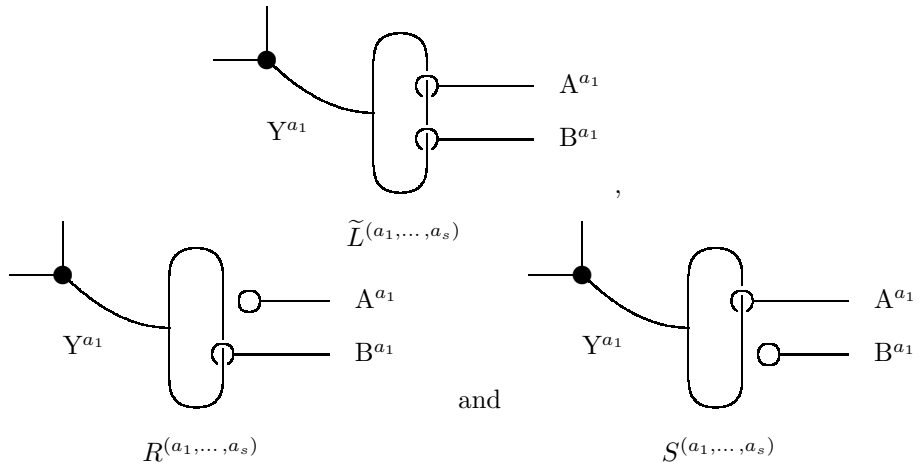
Now, Lemma 2.4.1 indicates that

$$\Sigma_{K^L}^p - \Sigma_K^p = \sum_{(a_1, \dots, a_s) = (1, \dots, 1)}^{(p, \dots, p)} [\Sigma_K^p, \tilde{L}^{(a_1, \dots, a_s)}],$$

with identical expressions holding when  $L$  is replaced by  $L'$  or  $L''$ . The lemma in question then follows from the following equation, which we will subsequently demonstrate:

$$(3.3.1) \quad [\Sigma_K^p, \tilde{L}^{(a_1, \dots, a_s)}] = [\Sigma_K^p, \tilde{L}'^{(a_1, \dots, a_s)}] - [\Sigma_K^p, \tilde{L}''^{(a_1, \dots, a_s)}].$$

Consider the mixed Y-link  $\tilde{L}^{(a_1, \dots, a_s)}$  corresponding to some  $s$ -tuple  $(a_1, \dots, a_s)$ . Let the mixed Y-links  $R^{(a_1, \dots, a_s)}$  and  $S^{(a_1, \dots, a_s)}$  be obtained from it by modifying it in the included subspace  $W_{\tilde{F}}^{a_1}$  as follows:



Lemma A.0.6, indicates that

$$[\Sigma_K^p, \tilde{L}^{(a_1, \dots, a_s)}] = [\Sigma_K^p, R^{(a_1, \dots, a_s)}] + [\Sigma_K^p, S^{(a_1, \dots, a_s)}].$$

Equation 3.3.1 follows from the identifications

$$(3.3.2) \quad [\Sigma_K^p, R^{(a_1, \dots, a_s)}] = -[\Sigma_K^p, \widetilde{L}''^{(a_1, \dots, a_s)}],$$

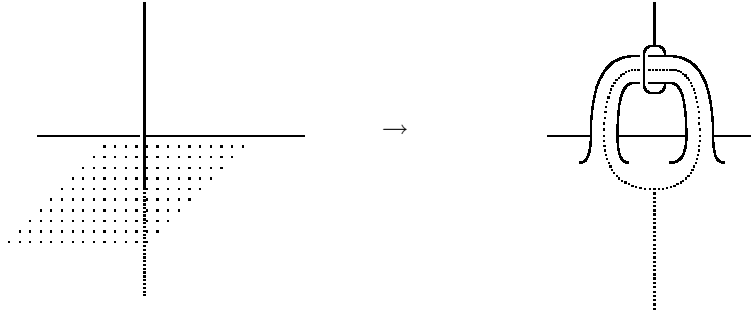
$$(3.3.3) \quad [\Sigma_K^p, S^{(a_1, \dots, a_s)}] = [\Sigma_K^p, \widetilde{L}'^{(a_1, \dots, a_s)}].$$

We leave the reader to check these relations. The identifications proceed by removing claspers when they have a leaf that bounds a disk in the complement of the rest of the graph. Note that the half-twist in  $B^{a_1}$  is smoothed at the expense of the minus one (using Lemma A.0.2).

□

**3.4. Proof of Lemma 2.0.12.** Choose a Seifert surface  $F$  for  $K$ , and an orientation for  $K$ , and hence an orientation for the Seifert surface. Represent this orientation by a normal vector field. Denote the embedded trivalent graph that is associated to a position of  $L$  that only meets this Seifert surface in transversal intersections of internal edges, by  $E_L$ .

Let  $L$  be used again (abusing the notation) to denote the mixed Y-link that results from the modifications of this embedding with respect to this Seifert surface that are described in Section 2.2. In this case, there are no legs to worry about, so the only modifications occur when edges of the associated clasper graph intersect the Seifert surface.



The mixed Y-link  $L$  comprises of  $s$  Y-components (order them  $Y_1, \dots, Y_s$ ) and a number of other claspers, in the complement of some Seifert surface  $\tilde{F}$ .

Recall that Lemma 2.4.1 indicates that

$$\Sigma_{KL}^p - \Sigma_K^p = \sum_{(a_1, \dots, a_s) = (1, \dots, 1)}^{(p, \dots, p)} [\Sigma_K^p, \widetilde{L}^{(a_1, \dots, a_s)}],$$

using notation explained there.

Let us consider one of these terms, corresponding to some  $p$ -tuple  $(a_1, \dots, a_s)$ . We will show that corresponding to each edge of  $E_L$ , to whose endpoints the two Y-components  $Y_i$  and  $Y_j$  are associated, there will be an equation in  $\mathbb{Z}_p$  relating  $a_i$  and  $a_j$  which must be satisfied in order that the corresponding term  $[\Sigma_K^p, \widetilde{L}^{(a_1, \dots, a_s)}]$  be non-zero.

Orient the edges of  $E_L$ . Define functions  $i$  and  $f$  from the edges of  $E_L$  to the labels of the Y-components of  $L$  so that  $Y_{i(e)}$  corresponds to the origin of the edge  $e$ , and the Y-component  $Y_{f(e)}$  corresponds to its end. Let  $\langle e, F \rangle$  denote the signed sum of intersections of the edge  $e$  (oriented from  $i(e)$  to  $f(e)$ ) with  $F$  (where a plus is an intersection in the direction of the orienting normal vector field).



**Lemma 3.4.1.**

If for the  $s$ -tuple  $(a_1, \dots, a_s)$ ,

$$(3.4.1) \quad a_{f(e)} \neq a_{i(e)} + \langle e, F \rangle \in \mathbb{Z}_p,$$

for some edge  $e$  of  $E_L$ , then

$$[\Sigma_K^p, \tilde{L}^{(a_1, \dots, a_s)}] = 0.$$

Proof

Take such a pair of Y-components in  $L$ ,  $Y_{i(e)}$  and  $Y_{f(e)}$ . Every time the corresponding edge of  $E_L$  intersected the Seifert surface  $F$  it was broken into a clasp. Thus, in general,  $Y_{i(e)}$  and  $Y_{f(e)}$  will be linked in  $L$  via some chain of clasps. Denote them  $C_1$  up to  $C_\mu$  where a leaf of  $C_1$  links a leaf of  $Y_{i(e)}$ , with its other leaf linking a leaf of the clasper  $C_2$  etc. Denote the copies of each of these clasps that occur in the graph  $\tilde{L}^{(a_1, \dots, a_s)}$  by affixing a  $\mathbb{Z}_p$ -valued superscript.

To show this equation, we proceed to remove copies of the  $C_j$  when they have a leaf that bounds a disc (in the complement of the rest of the mixed Y-link is to be understood when we use this phrase). If, finally, we are left with a mixed Y-link where one of the Y-components has a leaf that bounds a disk, then we know that the contribution  $[\Sigma_K^p, \tilde{L}^{(a_1, \dots, a_s)}]$  will be zero.

If the edge  $e$  has no (resp. 1 positive, resp. 1 negative) intersection with the Seifert surface  $F$ , then the term  $[\Sigma_K^p, \tilde{L}^{(a_1, \dots, a_s)}]$  will be zero unless  $a_{f(e)} = a_{i(e)}$  (resp.  $a_{f(e)} = a_{i(e)} + 1$ , resp.  $a_{f(e)} = a_{i(e)} - 1$ ), for otherwise the Y-components will have leaves bounding discs.

Consider, then, the situation where the edge  $e$  does intersect  $F$  at at least two points, so that some extra clasps arise. If the first intersection is a positive (resp. negative) intersection, then we can remove all copies of  $C_1$  except the copy  $C_1^{a_{i(e)}+1}$  (resp.  $C_1^{a_{i(e)}-1}$ ). Proceeding, we remove all copies of  $C_2$  except the one that intersects the surviving copy of  $C_1$ . And so on. One can check that as a result of this process, the surviving copy of  $C_\mu$  will intersect precisely the Y-component  $Y_j^{a_{i(e)}+\langle e, F \rangle}$ .

□

**Lemma 3.4.2.**

If the  $s$ -tuple  $(a_1, \dots, a_s)$  satisfies the equations

$$(3.4.2) \quad a_{f(e)} = a_{i(e)} + \langle e, F \rangle \in \mathbb{Z}_p,$$

for every edge  $e$  of  $E_L$ , then

$$[\Sigma_K^p, \tilde{L}^{(a_1, \dots, a_s)}] = (\Sigma_K^p)^{\lambda'_L} - \Sigma_K^p \in G_s^Y \mathbf{M}.$$

Proof.

Take the mixed Y-link  $L$  that is obtained following Section 2.2, where every intersection with a Seifert surface is broken into a clasp. Orient each Y-component and clasper of  $L$ . This introduces an orientation on each Y-component and clasper of  $\tilde{L}^{(a_1, \dots, a_s)}$ . Forget every clasper with a leaf that bounds a disc (as in the previous proof).

Note that the linking numbers between pairs of bounding circles in  $L$  and the corresponding pairs in  $\tilde{L}^{(a_1, \dots, a_s)}$  are the same. Thus, the theorem follows from Lemma 3.1.2 and Lemma 3.1.3.  $\square$

It is clear that the Equations 3.4.2 are precisely the requirement that every loop on  $E_L$  have zero mod- $p$  linking number with  $K$ . Note that when at least one solution exists, there are precisely  $p$  solutions, corresponding to the possible values of  $a_1$ .  $\square$

#### 4. PART 2: THE RELATION TO VASSILIEV THEORY

In Part 2 we consider some very general questions about the relationship with Vassiliev theory of knot invariants obtained by composing finite-type three-manifold invariants with the branched cyclic covering construction.

The Vassiliev theory of finite type invariants of oriented knots in the three-sphere centers on a filtration of  $K(S^3)$ , the free abelian group generated by ambient isotopy classes of oriented knots:

$$K(S^3) \supset F_1 K \supset F_2 K \supset \dots$$

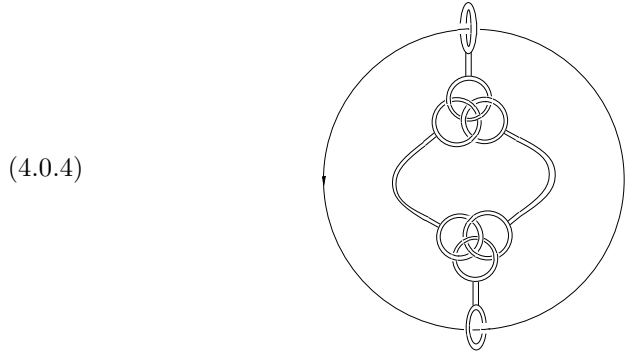
Our first speculation might be that any finite-type three-manifold invariant  $v$ , composed with  $\Sigma^p$ , is a finite-type invariant of knots. To begin, then, we will see that this hypothesis can be ruled out using simple arguments (for any choice of  $p$ ). The method of proof also provides some other interesting information: that the function  $|H_1(\Sigma_K^p)|$  can be unbounded on a sequence of knots of strictly increasing  $n$ -triviality.

**Theorem 4.0.3.** *Fix a positive integer  $p$ . For every integer  $n > 0$ , there exists a non-trivial element  $E_n \in F_n K$  such that*

$$(4.0.3) \quad \Sigma^p(E_n) \notin F_1^Y M.$$

Proof.

Let  $\Omega_2$  be the knot obtained from surgery on the following decoration of the unknot by a 0-framed surgery link (that is, the link obtained by replacing twice-punctured discs by surgery pairs, as in Appendix A). Similarly let  $\Omega_n$  denote the obvious extension to a knot obtained from surgery on a wheel with  $n$  spokes.



$\Omega_n$  is  $(n-1)$ -trivial, being obtained from surgery on a graph Y-link of degree  $n$  (alternatively we can identify it as a Kanenobu-Ng wheel [Kan, Ng]).

Our theorem will follow if we can identify an  $n' \geq n$  so that  $\Sigma^p(\Omega_{n'})$  is not an integral homology three sphere. Then the proof would be completed by setting  $E_n = \Omega_{n'} - U$ , where  $U$  is the unknot (see Matveev's theorem, Theorem A.0.1).

To this end we can exploit a formula for the order of the first homology of the  $p$ -fold branched cyclic covering of a knot  $K$  in terms of its Alexander polynomial  $A_K(t)$  [G, F, HK]. (The symbol  $|H_1(\Sigma^p(K))|$  below is extended to be zero if  $\Sigma^p(K)$  has non-vanishing first Betti number).

$$|H_1(\Sigma^p(K))| = \prod_{q=0}^{p-1} |A_K(e^{2\pi i \frac{q}{p}})|.$$

In [K] we calculated (alternatively, (do the work to) identify this as one of the knots considered in [Kan])

$$A_{\Omega_n}(t) = (1 - (1-t)^n)(1 - (1-t^{-1})^n),$$

so that

$$|H_1(\Sigma^p(\Omega_n))| = \prod_{q=0}^{p-1} |(1 - (1 - e^{2\pi i \frac{q}{p}})^n)|^2.$$

Define a function  $f(p, n)$  to be  $|H_1(\Sigma^p(\Omega_n))|$ . We are interested in the behaviour of  $f(p, n)$  as  $n$  goes to infinity whilst  $p$  is fixed.

Consider a factor  $(1 - \alpha(p, q)^n)$  in the above, where  $\alpha(p, q) = 1 - e^{2\pi i \frac{q}{p}}$ . If  $|\alpha(p, q)| < 1$ , then  $\lim_{n \rightarrow \infty} |1 - \alpha(p, q)^n| = 1$ . Alternatively, if  $|\alpha(p, q)| > 1$ , then  $\lim_{n \rightarrow \infty} |1 - \alpha(p, q)^n| = \infty$ . Note, however, that there are precisely two possible choices of  $\frac{q}{p}$  such that  $|\alpha(p, q)| = 1$ ; namely  $\frac{q}{p} = \pm \frac{1}{6}$  (which will occur in  $f(p, q)$  whenever  $6|p$ ). In these cases  $\alpha(6, \pm 1) = e^{\mp 2\pi i \frac{1}{6}}$ , and  $\lim_{n \rightarrow \infty} (1 - \alpha(6, \pm 1)^n)$  does not exist. We will get around this by choosing a subsequence; namely  $\lim_{n \rightarrow \infty} (1 - \alpha(6, \pm 1)^{6n+3}) = 2$ .

Finally, observe that there is always at least one factor  $|1 - \alpha(p, q)|$  such that  $|\alpha(p, q)| > 1$ . Then

$$\lim_{n \rightarrow \infty} f(p, 6n+3) = \infty.$$

□

See [Gor] for similar techniques.

We next consider a measure of the independence of these theories. The following theorem is modelled on Stanford-Trapp's definition of an invariant which is "independent of finite-type invariants", although note that the stated property (appears to be) substantially weaker.

It says that we may realise (some vector proportional to) any combination of primitive diagrams as the first non-vanishing term of the LMO invariant on the  $p$ -fold branched cyclic covering branched over a knot which is  $n$ -trivial, *where the  $n$ -triviality is a free parameter*.

According to the notation of [LeGr],  $\hat{Z}^{LMO}$  denotes the LMO invariant normalised by powers of the order of the first homology (the normalisation that satisfies the simple connect-sum formula).

**Theorem 4.0.4.** *Take positive integers  $n, p$  and  $s$ , and a knot  $K$  in  $S^3$  such that the  $p$ -fold branched cyclic covering of  $S^3$  branched over  $K$  is a rational homology*

three-sphere. Then, there exists a non-zero integer  $\alpha(s, n, p)$  such that for every  $\mathbb{Z}$ -linear combination of primitive degree  $s$  uni-trivalent diagrams on an empty skeleton,  $D$ , there exists a knot  $K(s, n, p, D)$  satisfying:

- $K(s, n, p, D)$  is  $n$ -equivalent to  $K$ ,
- $\hat{Z}^{LMO}(\Sigma_{K(s, n, p, D)}^p) - \hat{Z}^{LMO}(\Sigma_K^p) = p\alpha(s, n, p)D + \text{terms of higher order.}$

See Example 4.1.2 for an illustration of the simple idea behind this theorem.

*Remark 4.0.5.* In some sense, leaving the  $n$ -triviality of the realising knot a free parameter is the hard part of the above theorem. It is likely, for example, that such a realisation theorem (with  $n$  fixed at  $s$ ) is an extension of Garoufalidis's investigations of the LMO invariant on branched cyclic covers over doubled knots.

Note that if the  $n$ -triviality is not required to be free above, and  $p > 2$ , then a realisation with  $\alpha = 1$  exists. Can we find an improvement of the above theorem in the stated generality where  $\alpha(s, n, p) = 1$ ?

A corollary of Theorem 4.0.4 is:

**Corollary 4.0.6.** *Take a positive integer  $p$ . If  $v$  is a rational valued three-manifold invariant factoring non-trivially through the LMO invariant on integral homology spheres, then  $v \circ \Sigma^p$  is not a finite-type invariant of knots.*

The proof of this corollary is indicated in Section 4.3.

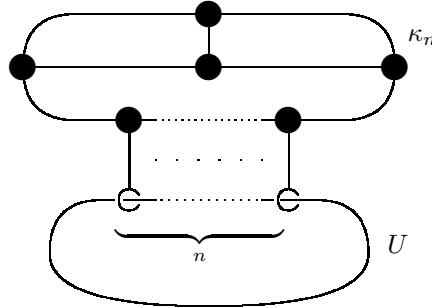
**4.1. Graph Y-links and the LMO invariant.** Graph Y-links in rational homology three-spheres realise particular first non-vanishing terms of the LMO invariant. It is convenient here to cite a direct calculation [LeGr]. Below, let  $\hat{Z}^{LMO}$  denote the version of the LMO invariant normalised by powers of the order of the first homology, following the notation of [LeGr].

**Theorem 4.1.1.** *Let  $M$  be a rational homology three-sphere and let  $L$  be a connected graph Y-link in  $M$  with some associated uni-trivalent diagram  $D_L$  (i.e. represented by the underlying graph of  $L$  with some choice of orientation at each vertex). Then,*

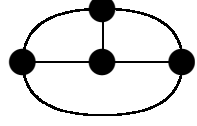
$$(4.1.1) \quad \hat{Z}^{LMO}(M^L) - \hat{Z}^{LMO}(M) = \pm D_L + \text{terms of higher order.}$$

Using Lemma A.0.2, one can always realise the opposite sign by introducing a half-twist.

**Example 4.1.2.** Before we consider the details of the realisation theorem, let us consider an example to highlight the simple idea involved. Consider the following decoration of the unknot  $U$  by a graph Y-link  $\kappa_n$ :



According to Theorem 2.0.8,  $\Sigma_{U^{\kappa_n}}^p - \Sigma_U^p$  lies in  $F_4^Y M$ , and moreover is proportional to the difference:



$$- S^3 \in G_4^Y M$$

To calculate the proportionality constant, we follow the procedure described in Example 2.0.15. Start by introducing a variable for each leg  $(\epsilon_1, \dots, \epsilon_n)$ . If  $\epsilon_j$  is zero, then that leg is to be sawn off; if it is 1 then that leg is sawn off with an extra wrap around the knot (see Example 2.0.15).

The underlying trivalent graph has three cycles. Two of the cycles are unlinked from the knot for every  $n$ -tuple  $(\epsilon_1, \dots, \epsilon_n)$ . The linking number of the third cycle with the knot will be  $\epsilon_1 + \dots + \epsilon_n$ .

Following Example 2.0.15, letting

$$F(n, t) = \sum_{(\epsilon_1, \dots, \epsilon_n) = (0, \dots, 0)}^{(1, \dots, 1)} (-1)^{\epsilon_1 + \dots + \epsilon_n} t^{\epsilon_1 + \dots + \epsilon_n} = (1 - t)^n,$$

then

$$\Sigma_{U^{\kappa_n}}^p - S^3 = \left( \sum_{q=0}^{p-1} F\left(n, e^{\frac{2\pi i q}{p}}\right) \right) ((S^3)^{\lambda_{\kappa_n}} - S^3) \in G_4^Y M.$$

Thus

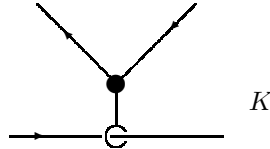
$$\hat{Z}^{LMO}(\Sigma_{U^{\kappa_n}}^p) = 1 + \epsilon \left( \sum_{q=0}^{p-1} (1 - e^{\frac{2\pi i q}{p}})^n \right) \kappa + \text{terms of higher degree},$$

for some  $\epsilon = \pm 1$ .

Given a knot  $K$ , an oriented trivalent diagram  $D$ , and a positive integer  $n$ , we choose a knot  $K_{(D,n)}$  as follows. Select a graph  $Y$ -link  $L$  in the three-sphere so that

$$\hat{Z}^{LMO}((S^3)^L - S^3) = D + \text{terms of higher degree}.$$

Then, locate  $L$  in a ball disjoint from  $K$  in the three-sphere. Select some edge of  $L$ , give it an orientation, and add  $n$  legs joining that edge to the knot  $K$  in that ball, so that each edge is added according to the following orientation (but otherwise, in any fashion):



There are many choices in this definition. Nevertheless:

**Lemma 4.1.3.**

$$\hat{Z}^{LMO}(\Sigma_{K_{(D,n)}}^p - \Sigma_K^p) = \left( \sum_{q=0}^{p-1} F\left(n, e^{\frac{2\pi i q}{p}}\right) \right) D + \text{terms of higher degree}.$$

This follows from the same logic as in the above example. It is clear that if  $D$  has surplus  $s$ , then  $K_{(D,n)}$  is obtained from  $K$  by surgery on a graph Y-link of degree  $\frac{s}{2} + n$ , and so is  $(\frac{s}{2} + n)$ -equivalent to  $K$  [Hab].

**4.2. Proof of Theorem 4.0.4.** Without loss of generality, let the combination to be realised be written  $S = \sum_{i=1}^t D_i$ . Take some positive integer  $l$ . Let  $K_{(S,l)}$  denote a knot:

$$(\dots((K_{(D_1,l)})(D_2,l))\dots)_{(D_t,l)}.$$

It is clear that

$$\hat{Z}^{LMO}(\Sigma^p(K_{(S,l)} - K)) = \left( \sum_{q=0}^{p-1} F\left(l, e^{\frac{2\pi i q}{p}}\right) \right) S + \text{terms of higher order}.$$

The number of legs to be added,  $l$ , is a free parameter, so that one can make the knot  $K_{(S,l)}$   $n$ -equivalent to  $K$  for arbitrarily large  $n$ .

The remaining difficulty is covered by the following lemma.

□

**Lemma 4.2.1.**

$$\lim_{l \rightarrow \infty} \left( \sum_{q=0}^{p-1} F\left(l, e^{\frac{2\pi i q}{p}}\right) \right) \neq 0.$$

Proof

Let  $\omega = e^{\frac{2\pi i}{p}}$ . Define a function  $f(l) = \sum_{q=0}^{p-1} F(l, \omega^q)$ . Take some positive integer  $l$ . We will show that  $f(l')$  is non-zero for at least one  $l'$  such that  $l \leq l' < l + p$ .

We assume that  $f(l+i) = 0$  for  $0 \leq i < p$ , and derive a contradiction. Now, we have assumed that  $f(l) = 0$ , so that, letting  $T_q = (1 - \omega^q)$ ,

$$f(l+1) = \sum_{q=0}^{p-1} (1 - \omega^q) T_q^l = - \sum_{q=0}^{p-1} \omega^q T_q^l = 0.$$

Proceeding with this assumption, one finds that

$$f(l+j) = (-1)^j \sum_{q=0}^{p-1} \omega^{jq} T_q^l = 0.$$

In other words, under this assumption, we have a matrix equation:

$$\begin{pmatrix} f(l) \\ -f(l+1) \\ f(l+2) \\ \vdots \\ \vdots \\ \vdots \\ (-1)^{p-1} f(l+p-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \dots & \omega^{p-1} \\ 1 & \omega^2 & \omega^4 & \dots & \dots & \omega^{2(p-1)} \\ \vdots & \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \dots & \vdots \\ \vdots & \vdots & \vdots & \dots & \dots & \vdots \\ 1 & \omega^{(p-1)} & \omega^{2(p-1)} & \dots & \dots & \omega^{(p-1)(p-1)} \end{pmatrix} \begin{pmatrix} T_0^l \\ T_1^l \\ T_2^l \\ \vdots \\ \vdots \\ \vdots \\ T_{p-1}^l \end{pmatrix}$$

The vector on the right is clearly non-zero, and the matrix is a Vandermonde matrix, and is non-singular, so the vector on the left cannot be zero. This is our contradiction.

□

#### 4.3. Proof of Corollary 1.0.1.

This is equivalent to finding for *any* vector  $D \in A_s$  an element  $E_{(D,n)} \in \mathbb{F}_n K$  such that

$$Z^{LMO}(\Sigma^p(E_{(D,n)})) = D + \text{higher order terms.}$$

This follows easily from the Theorem 4.0.4 (freely multiplying by rationals) and the observation that for knots  $K$  and  $L$ :

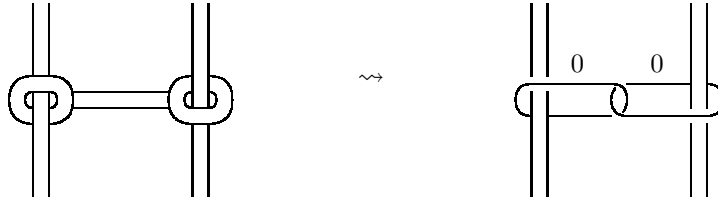
$$(4.3.1) \quad \Sigma^p(K \# L) = \Sigma^p(K) \# \Sigma^p(L).$$

□

### APPENDIX A. THE GOUSSAROV-HABIRO THEORY OF FINITE-TYPE INVARIANTS OF THREE-MANIFOLDS

The theory to be recalled in this section is due, independently, to Goussarov [G3] and to Habiro [Hab]. The original finite type theory (in the setting of  $\mathbb{Z}HS^3$ s) is due to Ohtsuki [Oht]; in that setting there are also definitions due to Garoufalidis and Levine [GL]; and the first definition to consider the set of all three-manifolds was due to Cochran and Melvin [CM]. The Goussarov-Habiro theory is defined in terms of a move on three-manifolds due to Matveev [Mat], closely related to a move of Murakami and Nakanishi's, on links [MN]. Our summary will be concise: we expect expositions of this theory to appear in the future ([GGP]).

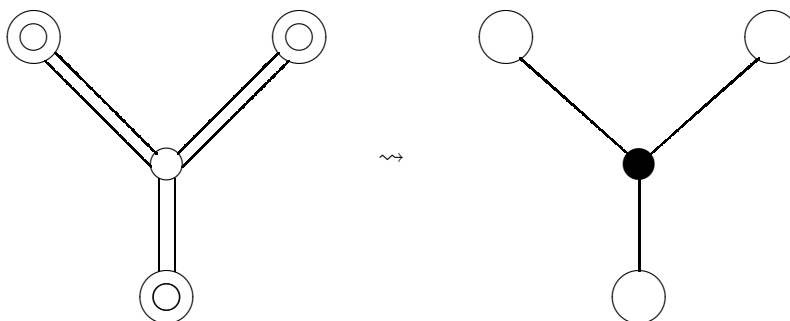
A *clasper* is an embedding of a standard band-summed pair of annuli into a three-manifold. The annuli are called the *leaves* of the clasper. This gives placement information for a two-component framed link, as follows, and a *move* on that clasper replaces that manifold with the manifold obtained by doing surgery on that link. The tubes in the diagram below can contain part of the surgery link presenting the three-manifold, and any other objects that may be embedded in the three-manifold:



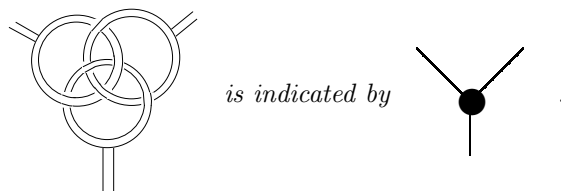
In this work, claspers in  $S^3$  will be depicted via blackboard framed diagrams. That is, circles should be thickened to annuli, and the other arcs should be thickened to bands, in the plane of the diagram. The following symbols will be used to indicate half-twists of bands:



A *Y-component* is an embedding of a standard triple of annuli band-summed into a disk, into a three-manifold. This gives placement information for a six-component surgery link. A *move* on that Y-component replaces that manifold with the manifold obtained by doing surgery on that link. This is Matveev's Borromeo move and Murakami-Nakanishi's Delta-unlinking move. It is convenient, here, to build a standard Y-component from three claspers, as follows:



where the following convention has been used:



It can be shown that this move generates an equivalence relation on the set of closed oriented three-manifolds.

**Theorem A.0.1** ([Mat]). *For two closed oriented three-manifolds  $M$  and  $N$ , there is an isomorphism  $H_1(M) \rightarrow H_1(N)$  preserving the linking form on the torsion if and only if  $N$  is obtained from  $M$  via a finite sequence of Y-moves.*

The Goussarov-Habiro theory is based on a filtration of  $\mathbf{M}$ , the free abelian group generated by homeomorphism classes of closed oriented three-manifolds.

The definition uses a *Y-link* in a three-manifold, which is a collection of disjointly embedded Y-components. Take a  $\mu$ -component Y-link  $L$ , and let  $M_L$  denote the three-manifold obtained by performing the moves on the Y-components of  $L$ . Define a vector  $[M, L] \in \mathbf{M}$  corresponding to a pair of some  $M$  and some Y-link  $L$  in  $M$  by

$$[M, L] = \sum_{L' \subset L} (-1)^{\#L - \#L'} M_{L'},$$

where the sum is over all Y-sublinks of  $L$  (so that there will be  $2^\mu$  such).



**Definition A.0.2.** Define the subspace  $F_n^Y M \subset M$  to be the subspace spanned by all vectors  $[M, L]$  with  $L$  an  $n$ -component Y-link.

It is clear that this filters  $M$ ,

$$M \supset F_1^Y M \supset F_2^Y M \supset F_3^Y M \dots,$$

and that a theory of finite-type invariants of three-manifolds can be constructed by declaring an invariant to be finite-type of order  $\geq n$  if it vanishes on the subspace  $F_{n+1}^Y M$ .

The associated graded quotients  $\frac{F_n^Y M}{F_{n+1}^Y M}$ , denoting them  $G_n^Y M$ , are finite dimensional, with a finite spanning set of generators of the form  $[M, L]$  where  $L$  is a *graph Y-link*. These are Y-links where every leaf of every Y-component links precisely one other leaf in a ball, as follows:



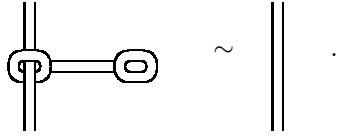
or is some standard link fixed to represent some element of homology. The full class will not occur in this work, so when we refer to a graph Y-link, we will mean specifically a Y-link where every leaf meets another in a ball.

To a graph Y-link is associated a uni-trivalent graph by taking a trivalent vertex for every Y-component, and joining edges when the associated leaves link in a ball. If the graph associated to some graph Y-link is connected, call the associated graph Y-link *connected*.

**Lemma A.0.3.** *If  $L$  is a connected graph Y-link, and  $L'$  is a Y-sublink of  $L$  not  $L$ , then  $M_{L'} \simeq M$ . A consequence is that*

$$[M, L'] = 0.$$

This follows from the fact that any clasper or Y-component with a leaf that bounds a disc in the complement of the rest of the Y-link, may be removed without affecting the result:



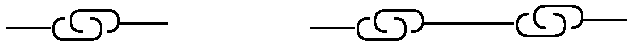
In the remainder of this section, we describe some aspects of the manipulation of clasper graphs that will be useful in ensuing calculations.

In this work, we will encounter situations where it is convenient to present a Y-link by decorating some other Y-link with a collection of claspers: the desired Y-link being recovered by surgery on those claspers. So, on occasions when precision is required, we will call such a link a *mixed Y-link*. The definition of the vector  $[M, L]$  is extended to mixed Y-links, taking the alternating sum over the Y-components

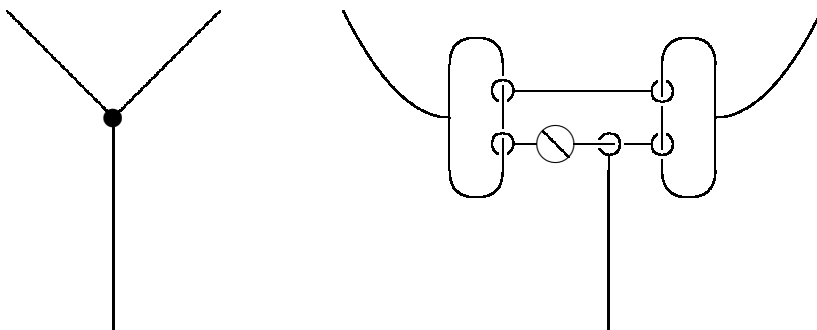
only. A *sublink* of a mixed link, is obtained by forgetting a number of Y-components or claspsers.

Now we collect some moves that will be useful in this paper.

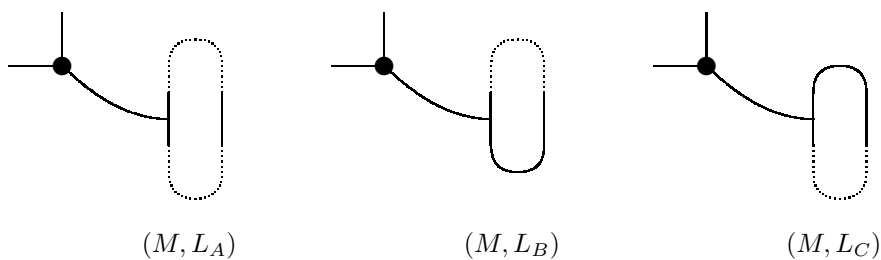
**Lemma A.0.4.** *The result of surgery on two mixed Y-links which differ in a ball as follows is the same.*



**Lemma A.0.5.** *The result of surgery on two mixed Y-links which differ in a ball as follows is the same.*

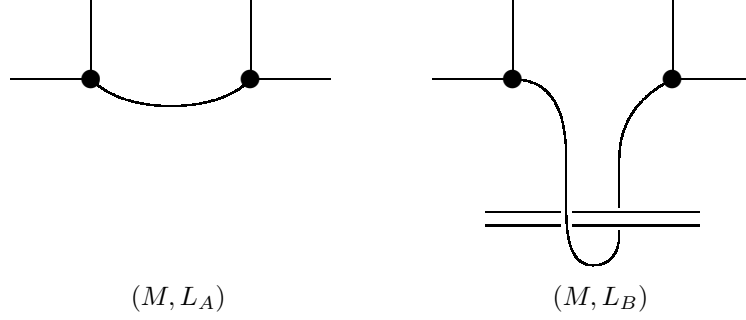


**Lemma A.0.6.** *Let  $L_A$ ,  $L_B$  and  $L_C$  be  $n$ -component Y-links that differ in a ball as follows. The dashed part of the leaf indicates that that part follows some path in the three-manifold before returning to the ball in question.*



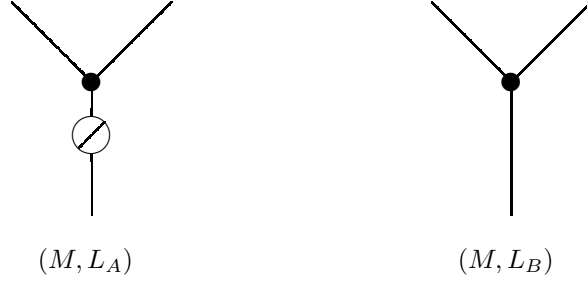
$$[M, L_A] = [M, L_B] + [M, L_C] \in G_n^Y \mathbf{M}.$$

**Lemma A.0.7.** *Let the Y-links  $L_A$  and  $L_B$  differ as follows. The tube can contain parts of surgery components or part of the rest of the graph:*



$$[M, L_A] = [M, L_B] \in \mathbf{G}_n^Y \mathbf{M}.$$

Finally, let  $L_A$  and  $L_B$  differ in a ball as follows:



**Lemma A.0.8.**

$$(A.0.2) \quad [M, L_A] = -[M, L_B] \in \mathbf{G}_n^Y \mathbf{M}.$$

**A.1. n-equivalence.** The property of  $n$ -equivalence was introduced by Ohya in the setting of knots [Ohy]; its importance for Vassiliev theory was observed by Goussarov [G1].

If two three-manifolds are  $n$ -equivalent then their difference lies in  $\mathbf{F}_{n+1}^Y \mathbf{M}$  and on this pair all finite-type invariants of order less than or equal to  $n$  agree.

**Definition A.1.1.** A  $n+1$ -scheme for  $M$  is a mixed Y-link  $L$  in  $M$  together with a set of  $n+1$  disjoint Y-sublinks of  $L$ ,  $L_1$  up to  $L_{n+1}$ .

For some  $n+1$ -tuple  $\{i_1, \dots, i_{n+1}\}$ , where  $i_k$  is either 0 or 1, the notation  $L_{i_1, \dots, i_j}$  is used to denote the Y-link that is obtained by forgetting those sublinks whose associated index is 1.

**Definition A.1.2.** A three-manifold  $N$  is  $n$ -equivalent to a three-manifold  $M$  if  $M$  has an  $n+1$ -scheme  $\{L; L_1, \dots, L_{n+1}\}$  such that

- $M_{L_0, \dots, 0} \simeq N$ ,

- $M_{L_{i_1, \dots, i_{n+1}}} \simeq M$  for any other multiplet.

In such a situation we will say that  $\{L; L_1, \dots, L_{n+1}\}$  is an  $n+1$ -scheme *relating*  $N$  to  $M$ .

**Lemma A.1.3.** *If  $M$  is  $n$ -equivalent to  $N$ , then*

$$M - N \in \mathbf{F}_{n+1}^Y \mathbf{M}.$$

In such a situation we also have a nice expression for  $M - N$  in the graded space  $\frac{\mathbf{F}_{n+1}^Y \mathbf{M}}{\mathbf{F}_{n+2}^Y \mathbf{M}}$ .

To introduce this expression we need to be more specific with some notation. Denote the relating  $n+1$ -scheme in  $M$  by  $\{L; L_1, \dots, L_{n+1}\}$ . Let  $\sigma(i)$  be the function giving the number of  $Y$ -components of the  $Y$ -sublink  $L_i$ . Order the  $Y$ -components of each  $Y$ -sublink  $L_i$ . For an  $n+1$ -tuple  $(a_1, \dots, a_{n+1})$ , where  $1 \leq a_i \leq \sigma(i)$ , let  $L^{(a_1, \dots, a_{n+1})}$  be the mixed  $Y$ -link in  $M$  obtained by forgetting all  $Y$ -components of each of these  $Y$ -sublinks, except precisely one  $Y$ -component from each  $L_i$ : that is, from  $L_i$  choose  $a_i$ .

**Lemma A.1.4.**

$$N - M = \sum_{\substack{(\sigma(1), \dots, \sigma(n+1)) \\ (a_1, \dots, a_{n+1}) = (1, \dots, 1)}}^{(\sigma(1), \dots, \sigma(n+1))} [M, L^{(a_1, \dots, a_{n+1})}] \in \mathbf{G}_{n+1}^Y \mathbf{M}.$$

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DEPARTMENT OF MATHEMATICAL AND COMPUTING SCIENCE, TOKYO INSTITUTE OF TECHNOLOGY

*E-mail address:* `kricker@is.titech.ac.jp`